Exam questions

Reasoning Lectures

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1 Questions

1. Show that the De Morgan laws $A \wedge B \dashv\vdash \neg(\neg A \vee \neg B)$ are derivable in Classical Logic, e.g exhibit a derivation of the sequent $A \wedge B \vdash \neg(\neg A \vee \neg B)$ and a derivation of the sequent $\neg(\neg A \vee \neg B) \vdash A \wedge B$ in G.CL.

2. Only one of the previous sequents is derivable in Intuitionistic Logic: exhibit a derivation of it in G.IL.

3. Show that the sequent $A \to (B \wedge C) \vdash (A \to B) \wedge (A \to C)$ is derivable in Intuitionistic Logic using G.IL.

4. The sequent $A \multimap (B \otimes C) \vdash (A \multimap B) \otimes (A \multimap C)$ is not derivable in Intuitionistic Linear Logic. Find a variant of the previous sequent where one or more (sub)formulas appear under the scope of !, and such that it is derivable in Intuitionistic Linear Logic. Show a derivation of it in G.ILL.

2 Answers

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,\mathrm{w}_L}{A \wedge B \vdash A}\,\wedge_L}{A \wedge B, \neg A \vdash}\,\neg_L \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{B, A \vdash B}\,\mathrm{w}_L}{A, B \vdash B}\,\mathrm{e}_L}{A \wedge B \vdash B}\,\wedge_L}{A \wedge B, \neg B \vdash}\,\neg_L}{A \wedge B, \neg A \vee \neg B \vdash}\,\vee_L$$
$$\dfrac{A \wedge B, \neg A \vee \neg B \vdash}{A \wedge B \vdash \neg(\neg A \vee \neg B)}\,\neg_R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,\mathrm{w}_L \quad \dfrac{\dfrac{B \vdash B}{B, A \vdash B}\,\mathrm{w}_L}{A, B \vdash B}\,\mathrm{e}_L}{A, B \vdash A \wedge B}\,\wedge_R}{A \vdash \neg B, A \wedge B}\,\neg_R}{\vdash \neg A, \neg B, A \wedge B}\,\neg_R}{\vdash \neg A \vee \neg B, A \wedge B}\,\vee_R}{\neg(\neg A \vee \neg B) \vdash A \wedge B}\,\neg_L$$

2. The first proof in 1. is a proof in G.IL as well:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,\mathrm{w}_L}{A \wedge B \vdash A}\,\wedge_L}{A \wedge B, \neg A \vdash}\,\neg_L \quad \dfrac{\dfrac{\dfrac{B \vdash B}{B, A \vdash B}\,\mathrm{w}_L}{A, B \vdash B}\,\mathrm{e}_L}{A \wedge B \vdash B}\,\wedge_L}{A \wedge B, \neg B \vdash}\,\neg_L}{A \wedge B, \neg A \vee \neg B \vdash}\,\vee_L}{A \wedge B \vdash \neg(\neg A \vee \neg B)}\,\neg_R$$

3.

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad \dfrac{\dfrac{B \vdash B}{B, C \vdash B}\,\mathrm{w}_L}{B \wedge C \vdash B}\,\wedge_L}{A, A \to (B \wedge C) \vdash B}\,\to_L}{A \to (B \wedge C) \vdash A \to B} \qquad \dfrac{\dfrac{A \vdash A \qquad \dfrac{\dfrac{C \vdash C}{C, B \vdash C}\,\mathrm{w}_L}{B \wedge C \vdash C}\,\wedge_L}{A, A \to (B \wedge C) \vdash C}\,\to_L}{A \to (B \wedge C) \vdash A \to C}}{A \to (B \wedge C) \vdash (A \to B) \wedge (A \to C)}\,\wedge_R$$

4.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{!B \vdash B}\,\mathrm{d!}}{!B, !C \vdash B}\,\mathrm{!w}}{!B \otimes !C \vdash B}\,\otimes_L}{A, A \multimap (!B \otimes !C) \vdash B}\,\multimap_L}{A \multimap (!B \otimes !C) \vdash A \multimap B}\,\mathrm{d!}}{!(A \multimap (!B \otimes !C)) \vdash A \multimap B} \qquad \dfrac{\dfrac{\dfrac{A \vdash A \qquad \dfrac{\dfrac{\dfrac{C \vdash C}{!C \vdash C}\,\mathrm{d!}}{!C, !B \vdash C}\,\mathrm{!w}}{!B, !C \vdash C}\,\mathrm{e}}{!B \otimes !C \vdash C}\,\otimes_L}{A, A \multimap (!B \otimes !C) \vdash C}\,\multimap_L}{A \multimap (!B \otimes !C) \vdash A \multimap C}\,\mathrm{d!}}{!(A \multimap (!B \otimes !C)) \vdash A \multimap C}}{!(A \multimap (!B \otimes !C)), !(A \multimap (!B \otimes !C)) \vdash (A \multimap B) \otimes (A \multimap C)}\,\otimes_R}{!(A \multimap (!B \otimes !C)) \vdash (A \multimap B) \otimes (A \multimap C)}\,\mathrm{!c}$$

3 Gentzen sequent calculi

3.1 Sequent calculus G.CL (Classical Logic)

Axioms

$$\frac{}{p \vdash p}\,\mathrm{Id} \qquad \bot\,\frac{}{\bot \vdash} \qquad \frac{}{\vdash \top}\,\top$$

Logical Rules

$$\wedge_L\,\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\,\wedge_R \qquad \neg_L\,\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\,\vee_R \qquad \vee_L\,\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\,\neg_R$$

$$\rightarrow_L\,\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}\,\rightarrow_R$$

Structural Rules

$$\mathrm{w}_L\,\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\,\mathrm{w}_R \qquad \mathrm{c}_L\,\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,\mathrm{c}_R$$

$$\mathrm{e}_L\,\frac{\Gamma, B, A, \Gamma' \vdash \Delta}{\Gamma, A, B, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, B, A, \Delta'}{\Gamma \vdash \Delta, A, B, \Delta'}\,\mathrm{e}_R \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{cut}$$

3.2 Sequent calculus G.IL (Intuitionistic Logic)

Axioms

$$\frac{}{p \vdash p}\,\mathrm{Id} \qquad \bot\,\frac{}{\bot \vdash} \qquad \frac{}{\vdash \top}\,\top$$

Logical Rules

$$\wedge_L\,\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\wedge_R$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\vee_{R1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\vee_{R2} \qquad \vee_L\,\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}$$

$$\rightarrow_L\,\frac{\Gamma \vdash A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\,\rightarrow_R \qquad \neg_L\,\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\,\neg_R$$

Structural Rules

$$\mathrm{w}_L\,\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\,\mathrm{w}_R \qquad \mathrm{c}_L\,\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}$$

$$\mathrm{e}_L\,\frac{\Gamma, B, A, \Gamma' \vdash \Delta}{\Gamma, A, B, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash A \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{cut}$$

3.3 Sequent calculus G.ILL (Intuitionistic Linear Logic)

Axioms + 1_L

$$\frac{}{p \vdash p}\,\mathrm{Id} \quad \bot_L\,\frac{}{\Gamma, \bot \vdash A} \quad \frac{}{\Gamma \vdash \top}\,\top_R \quad \frac{}{\vdash 1}\,1_R \quad 1_L\,\frac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta}$$

Logical Rules

$$\wedge_{L1}\,\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge_{L2}\,\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\wedge_R$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\vee_{R1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\vee_{R2} \qquad \vee_L\,\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}$$

$$\otimes_L\,\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \qquad \frac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}\,\otimes_R \qquad \mathrm{d}!_L\,\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\mathrm{p}!_R$$

$$\multimap_L\,\frac{\Gamma \vdash A \qquad \Gamma', B \vdash \Delta}{\Gamma, \Gamma', A \multimap B \vdash \Delta} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,\multimap_R \qquad \sim_L\,\frac{\Gamma \vdash A}{\Gamma, \sim A \vdash} \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \sim A}\,\sim_R$$

Structural Rules

$$!\mathrm{w}\,\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad !\mathrm{c}\,\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \mathrm{e}\,\frac{\Gamma, B, A, \Gamma' \vdash \Delta}{\Gamma, A, B, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash A \qquad \Gamma', A \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta}\,\mathrm{cut}$$